In the following pages I will try to elucidate to what extent Frege’s logicism coincide with Benacerraf’s definition, and with Hempel’s own logicism. This last comparison will center on the problem of the correferentiality of definitions of number, as a possible way to analyze the “structuralism” of Frege as well as that of Dedekind.

I

Logicism (or the “logistic thesis”) is characterized in vol. 5 of Edwards’ Encyclopedia as “the claim that the theory of numbers is reducible to logic.” In vol. 3, p. 226, Michael Dummett is more explicit: “all arithmetical notions could be defined in terms of notions required for logic in general, and all arithmetical laws could be proved from principles likewise required.”

This seems to be an exact definition of Frege’s endeavors. But this is certainly not the only possible definition of logicism. Benacerraf adds to the definition above the requirement of “an epistemological claim about HOW THIS EXPLAINS THE A PRIORI CHARACTER OF ARITHMETIC” ([1], p. 35). Furthermore, the specific epistemological claim Benacerraf has in mind must be based on empiricist principles, and empiricism hardly goes well with Frege’s conception of Geometry as synthetic a priori.
Frege’s opinions about geometry have shocked more than one philosopher who can only blame Frege’s kantian background for what appears as a misplaced tribute to tradition. But Frege had strong reasons for this view which as a matter of fact, not only does not conflict with his ideas on arithmetic, but stems from his general conception of mathematics as much as his ideas on arithmetic do.

Frege was impressed, as everybody else, by the existence of non-euclidean geometries. We can talk about spaces of ten dimensions easily. It is true that we can not imagine them: we have only "the same old intuition of Euclidian space"; and yet we can go on and develop those strange geometries without inconsistency. But something is strange here. To develop these geometries we had to deny some axiom or other of, say, euclidean geometry. What kind of axiom is one which can be denied without inconsistency? We conclude that geometrical axioms are not “ever-present.” If they applied everywhere, their denial would lead *ipso facto* into contradiction. So, geometrical axioms are “local truths” (no pun intended). But then, to develop geometries we must go beyond ever-reaching principles, including, of course, analytic principles.

In paragraph 14 of the Grundlagen, Frege contrasts this lack of generality of geometrical truths with the scope of arithmetical truths, “the widest domain of all” including everything thinkable. Also, in “On Formal Theories of Arithmetic” (1885) we read that “we can count just about everything that can be an object of thought... What is required is really no more than, a certain sharpness of delimitation, a certain logical completeness” (p. 94, translation of E. W. Kluge). In contrast, “the axioms of geometry express the peculiarities of what is spatial” (id. p. 95). So Frege’s ideas on geometry are not the whimsical remnants of old-fashioned kantianism, but a thesis of his own about the relative scopes of geometrical axioms and arithmetical truths. (It is interesting to note that at the end of his life, after an all-too-late distrust of set theory, Frege reached the conclusion that arithmetic is synthetic *a priori* also. Whether this has something to do with a special “intuition” is another problem.)

Now, this leaves Frege with an important problem: where does the generality of arithmetic comes from? Well, if the scope of application of arithmetical truths is so large as to embrace “not only the actual, not only the intuitable, but everything thinkable”, as
J. L. Austin translates, then Frege can not think of other source of such tremendous generality but logic itself, the science of "whatever thinkable" par excellence. So, both the claim that arithmetic is all-embracing, and the claim that this generality comes from logic, receive simultaneous expression in the dictum that arithmetic is analytic. Analyticity will mean here not the kantian relation between the contents of subject and predicate but the property of being provable from general logical laws and the propositions involved in admissible definitions.

Leaving aside for the time being the thorny issue of the "admissible definitions", we must notice that we have here not only a reduction of arithmetic to logic as the quotes with which we started more or less demanded, but also the basis to satisfy Benacerraf's further requirement about an explanation of the a priori character of arithmetic. Nevertheless, such explanation depends not upon strong empiricist claims, but upon the characteristic of logical truths that also makes arithmetic different from geometry: generality.

Now, some people might bring into question whether logic itself has generality enough to communicate to arithmetic. Intuitionism, some polivalent logicians like Lukasiewicz, quantum logicians, some fuzzy, paraconsistent, relevant and free logicians, might question the exact scope of logic (or at least of Frege's logic), not to mention holistic theories like Quine's in "Two Dogmas..." which would admit changing scopes. Be it as it may, the fact is that according to Frege logical laws apply to anything thinkable and that it is from here that arithmetical truths receive their own generality. How would have Frege answered all these people questioning the domain of application of his system of logic? I think he would have said that logic carries its own justification on its sleeve and requires no more. As the turtle hinted to Achilles, how do you justify logic if it encompasses all?

Though late in his life Frege talked about a "logical source of knowledge" the details are precious scarce and mainly negative. The reason for his silence is easy to understand: because of its generality, any justification of logic would presuppose logic itself. In p. xvii of the introduction to the Grundgesetze, Frege wrote (in Furth's translation): "The question why and with what right we acknowledge a law of logic to be true, logic can answer only
bers because, according to him, these are objects. So, Frege must prove both that they exist, and also that there is only one of each. His solution is well known. First we define the notion of a property being exemplified \(n\) times. Having this property is "having" the number \(n\), so, in a certain sense, \(n\) is a property certain properties have. And since \(n\) is a monadic property, by extensionality it is common to read that \(n\) is the characteristic set of properties such that they have the property of having \(n\) instances.

But, is this really the number \(n\)?

III

Quine reflected in the curious fact that a natural number in Frege's theory is the class of all concepts equinumerous with any concept under which exactly that many objects fall, but in von Neumann's theory it is the class of all smaller natural numbers and in Zermelo's it is the class of just its predecessor. But if the numbers generated in any of these accounts has equal right to be called THE natural numbers, then really nothing qualifies as THE natural numbers: "That all are adequate as explanations of a natural number means that natural numbers, in any distinctive sense, do not need to be reckoned into our universe in addition" ([3], p. 263).

Quine views definitions as notational abridgments too (see [4], p. 82) but this is neutral with respect to any theory of real definitions. To define the natural numbers, any progression "will do nicely" ([3], p. 262). Benacerraf took up this suggestion and went on to argue that, since we can take number \(x\) to be a certain unique member in any given recursive progression but we do not have any reason to prefer one particular progression (classes of concepts, sets of sets, "representatives" of equivalence classes), then, since \(x\) can not be all those different objects at once, it is none; there is no such object \(x\). In other words, since it is okay to substitute it with any of these things, we can regard it as nothing but a word with a place-holder function. (Cf. [5], pp. 289-291.)

This position seems to conflict with the usual interpretation of Frege's platonism. Numbers are preexisting independent entities and Frege would not stand by this "anything goes" quinean attitude. Or would he?
Frege writes that numbers are "objects given directly to our reason and, as its nearest kin, utterly transparent to it" (Grundlagen, par. 105). This does not entail any subjectivity, just as it does not entail in the similar case of logic. Numbers are next of kin to our reason (as logical entities that they are) while at the same time maintaining total independence from us. "These objects are not subjective fantasies. There is nothing more objective than the laws of arithmetic" (Ib.). With regard to these matters Frege's thought has a clarity we will soon miss in Dedekind.

Now, are we going to conclude from these protests of objectivity that two correct definitions of a number must have the same reference? The step is tempting, but let us resist the temptation until we consider Hempel's notion of logicism.

In its main lines, Hempel's notion of logicism is that of Carnap or Frege. But we can detect in it a more pragmatic character of the mathematical constructs. In the pursuit of empirical knowledge, mathematics is, to use Hempel's own colorful words, "a theoretical juice extractor" ([6], p. 391).

When Hempel rejects Peano system for natural number as insufficient to capture the "customary meaning" of the number 2, we can feel we are facing a frontal rejection of the quinean "any isomorphism goes" attitude. But, as it turns out, the "customary meaning" is anything but customary.

Hempel does not want to capture some intuitive notion "everybody has in mind" about numbers. He wants a logical, not a psychological sense of "meaning". What he looks for is a logical reconstruction of the concepts of arithmetic "in the sense that if the definitions are accepted, then those statements in science and everyday discourse which involve arithmetical terms can be interpreted coherently and systematically in such a manner that they are capable of objective validation" ([6], p. 387). A number will be anything that gives the right results for our inferences. No correspondence mentioned. Now, is this Frege's own position?

An argument that could be used in this direction is offered by Dummett. He notes that because of Frege's theory that only in the context of a sentence does a name have a reference, Frege has to say that mathematical terms like "2" can only have reference inside mathematical statements. But if to know the truth conditions of the mathematical sentence "F(2)" is to know what "2"
refers to, then, to say that "2" denotes, that 2 exists, is simply to say that "F(2)" has determinate truth conditions. (Cf. [7], p. 212.)

In par. 60 of the Grundlagen, Frege explains that the self-substience of numbers we had noted before does not mean but that they are not predicates or attributes. It says nothing about the reference of "2" all by itself. W. Tait reads this as "to say that a term refers is to say that it is a meaningful term" ([8], p. 15). But if this is all there is to saying that numbers exist, and Frege is not committed to giving "real" definitions, what is there left to judge whether a definition is good except Hempel's pragmatic criterion of yielding the right results when applied?

And certainly there is a "pragmatic" ring to Frege's assertion in par. 70 that "Definitions show their worth by proving fruitful." It is not only that fruitfulness is a sure sign that the definition is correct; it is also a necessary condition: if a definition does not give anything new, it "should be rejected as completely worthless." How are we to harmonize these claims with the pure picture of definition-abridgment? Is he making just the innocent claim that definitions help us to be explicitly conscious of previously implicit knowledge?

Even if Frege had only in mind some kind of psychological fruitfulness, we also must deal with par. 107 of the Grundlagen, practically at the end of the book, where Frege tells us that in the definition of the Number which belongs to the concept F he does not attach any decisive importance "to bringing in the extensions of concepts at all." The only way to read this is as Frege's literal acceptance of alternative definitions. Now, this can receive at least two interpretations.

First: for Benacerraf "The moral is inescapable. Not even references needs to be preserved" ([1], p. 30). Frege suddenly is on Quine's side of the fence. Fregean Numbers would be "numbers," just in the sense of being isomorphic replacements with the right kind of formal properties. The question would not be What is 2?, but rather What could we say 2 is in order for mathematics to work?

A second possible interpretation does not saddle Frege with such drastic conceptions. In par. 67 Frege had questioned already the "necessity" of a particular definition. It was important that the same object had alternative possible definitions for two rea-
sons: In the first place, the definition is supposed to be saying nothing about the object, only about the symbol used to represent the object. No property of the object can compel us to regard a definition as "necessary" because we are not laying down a property of the object itself.

In the second place, Frege hints at his famous sense-denotation distinction and defends having different definitions as a special case of recognizing something again, "even although it is given in a different way." So, two definitions can be different in a couple of ways: they may refer to different objects, as Benacerraf would want it, or they may just be different in the way they present the object. If Frege is thinking of the latter possibility in par. 107, then he is defending the possibility of different definitions of the same object but not the possibility of non-correferential ones.

APPENDIX

In [9] p. 2, Hellman proposes that "A general structuralist outlook... can be found in embryonic form in Dedekind." Basically, structuralism comprises the deductive investigation of structural relationships in mathematics, without much concern with what the absolute identity of mathematical objects is. It is a free investigation in the sense that it does not need to comply with previous metaphysical intuitions as to what the ultimate components of mathematical reality are.

The problems we have encountered in Hempel's notion of customary meaning can reappear without much modification in a structuralist treatment if we are not worried with the question "Does this definition captures our previous intuitions"? Of course this seems to be a problem for any approach at a reconstruction of mathematics. We begin demanding from our numbers more than the verification of the right mathematical statements; as Russell says, we also want to have two eyes and one nose. A system in which "1" meant 100 "might be all right for pure mathematics, but would not suit daily life" ([10], p. 9). But after a while and the definition of 2 as the class of all couples, Russell adds: "At the expense of a little oddity this definition secures definiteness and indubitableness; and it is not difficult to prove that numbers so defined have all the properties that we expect numbers to have"
[10], p. 18). Definiteness is hardly a decisive factor if we are to choose between noncorreferential definitions. Indubitableness means just that we are assured that what is being offered as the reference of “2” exists, but if Frege, in the interpretation of Dummett, is right, it comes down to say that “F(2)” is meaningful. We are left with the third quality, having two eyes and one nose. But is not this just good old fruitfulness? Why could not two non-correferential definitions be equally fruitful?

In Dedekind we find the same recognition of the oddity of the purported references of the numbers. “I feel conscious that many a reader will scarcely recognize in the shadowy forms which I bring before him his numbers which all his life long have accompanied him has faithful and familiar friends” writes he in the preface to the first edition of WAS SIND… (translation by W. W. Beman).

Well, if numbers are not what we are used to think they are (if we think of them at all) then what are they? In par. 73 Dedekind offers the numbers as the result of a process of abstraction. We disregard everything except that they are different from each other and that they are ordered in an omega sequence. “With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind” he writes.

The passage is obscure. Is Dedekind saying that we literally create new entities, that we literally bring them into being, with all the ontological implications that carries? Or is he just saying that what is important in mathematics is only to pay attention to the abstract structures of the mathematical relations disregarding precisely any ontological inquiries? Was he endorsing in that passage conceptualism or structuralism?

The latter position is not only ontologically neutral; it has been used to argue against the necessity of postulating numbers as special entities, and furthermore to argue against their existence. This is the strategy of Benacerraf in [5], p. 291, where we read that “numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an abstract structure.” Whether Benacerraf is right or not in drawing this conclusion, the point here is that on the structuralist reading Dedekind could spouse diverse attitudes to-
wards what are the ontological commitments in the definition of natural numbers.

Tait on the other hand (and recently Kitcher in [11]) spous-es the creationist interpretation and concludes that the system of numbers has not only been created, but this creation has taken the form of specifying once and for all the grammar and logic of the system. How seriously do we have to take this creation? As Hellman remarks, to talk of creating numbers by a mental act is stronger than unqualified platonism, and certainly does not make it easier to read Dedekind consistently as a structuralist.

I would like to close this review with the suggestion that perhaps Dedekind was thinking of creation in an analogous way to when we say that a statue is created by “abstracting” it from the block of marble. When in par. 34 Dedekind speaks about the Representative of a class, with blissed disregard to problems like the axiom of choice. It is worth noting that the Representative is not literally created. It is just endowed with its quality through a deprivation of identifying characteristics; any other system could do as well. In a similar way, the numbers we freely create (that is, that we free from other characteristics) might be just fulfilling a function of representation. And they would only be “created” as we create a teenager out of a 14 years old kid.

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